Summary of topics

- Re-introduction to Predicate Logic
- Syntax of Predicate Logic
- Semantics of Predicate Logic
- Natural Deduction for Predicate Logic

Motivation

To show satisfiability, all you need to do is find an interpretation where the formula is true.

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Theorem

Show that $1 + 1 = 3$ *is satisfiable.*

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Theorem

Show that $1 + 1 = 3$ *is satisfiable.*

Proof.

Let $\mathcal{M}$ be

$$\begin{array}{lcl} \mathrm{dom}(\mathcal{M}) & = & \{1\} \\ (+)^{\mathcal{M}} & = & \{((1,1),1)\} \\ 1^{\mathcal{M}} & = & 1 \\ 3^{\mathcal{M}} & = & 1 \end{array}$$

Then $[\![1 + 1 = 3]\!]_{\mathcal{M}} = \mathtt{true}$ $\square$

Motivation

To show non-validity, all you need to do is find an interpretation where the formula is *false*.

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To show non-validity, all you need to do is find an interpretation where the formula is *false*.

Theorem

$1 + 1 = 3$ *is not valid.*

Proof.

Let $\mathcal{M}$ be any standard model of arithmetic. We have that

$$[\![1 + 1 = 3]\!]_{\mathcal{M}} = \texttt{false}$$

□

Motivation

For satisfiability and non-validity, we only needed to think about one interpretation. For validity, you need to think about *all* interpretations.

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For satisfiability and non-validity, we only needed to think about one interpretation. For validity, you need to think about *all* interpretations.

Natural deduction allows you to prove validity without thinking about interpretations at all.

Natural deduction for Predicate Logic

In this presentation of natural deduction, we only consider *sentences* (aka formulas with no free variables).

This simplifies the presentation, and the restriction is easy to work around: whenever you'd like a free variable, add a constant symbol to your vocabulary.

We'll use a, b, c for constants, t for terms, and x, y, z for variables.

Natural deduction for Predicate Logic

Inference rules for Propositional Logic $+$ seven rules for quantifiers and equality

Operator	**Introduction**	**Elimination**
$\forall$	$\forall$-I	$\forall$-E
$\exists$	$\exists$-I	$\exists$-E
$=$	$=$-I	$=$-E1 $=$-E2

Contexts

Suppose A is a sentence with one or more term-shaped holes (written $\cdot$). Then $A(t)$ is the result of filling all the holes with the term t.

Example

If A is the context $\cdot = a \wedge b = \cdot$ then

$$A(a) \text{ is } a = a \wedge b = a, \text{ and}$$

$$A(f(c)) \text{ is } f(c) = a \wedge b = f(c)$$

.

$=$ Introduction and Elimination

$=$-introduction: $$\frac{}{t = t}\ (=\text{-I})$$

"Every term equals itself"

$=$ Introduction and Elimination

$=$-introduction: $$\frac{}{t = t}\ (=\text{-I})$$

"Every term equals itself"

$=$-elimination (1): $$\frac{t = t' \qquad A(t)}{A(t')}\ (=\text{-E1})$$

$=$-elimination (2): $$\frac{t = t' \qquad A(t')}{A(t)}\ (=\text{-E1})$$

"Equal terms have the same properties"

Arbitrary constant symbols

A constant symbol is **arbitrary** if we haven't assumed anything about it—that is, if it does not occur in any undischarged assumption.

Intuitively: an arbitrary constant symbol can be assigned any element of the domain, and the formula will still hold.

$\forall$ Elimination

$\forall$-elimination:

$$\frac{\forall x A(x)}{A(t)} \; (\forall\text{-E})$$

"Any property that holds in general holds for any specific instance."

$\forall$ Introduction

$\forall$-introduction:

$$\frac{A(c) \qquad \begin{array}{c} c \text{ not free in A} \\ c \text{ is arbitrary} \end{array}}{\forall x A(x)} \; (\forall\text{-I})$$

"If we can prove a property about c without assuming anything about c, then the property holds in general."

$\forall$ Introduction

Without the "c is arbitrary" side condition, $\forall$-I would be completely broken.

Q: Why is the following argument wrong?

| Socrates is a man.
| All men are mortal.
|---
| Therefore, everybody is a man.

$\forall$ Introduction

Without the "c is arbitrary" side condition, $\forall$-I would be completely broken.

Q: Why is the following argument wrong?

| Socrates is a man.
| All men are mortal.
|---
| Therefore, everybody is a man.

A: Socrates is not arbitrary, so this $\forall$-I does not apply.

$\exists$ Introduction

$\exists$-introduction:

$$\frac{A(t)}{\exists x A(x)} \; (\exists\text{-I})$$

"If t satisfies A, then something satisfies A."

$\exists$ Elimination

$\exists$-elimination:

$$\dfrac{\exists x A(x) \qquad \begin{array}{c}[A(c)] \\ \vdots \\ B\end{array} \qquad \begin{array}{c} c \text{ not free in } A, B \\ c \text{ is arbitrary}\end{array}}{B}\ (\exists\text{-E})$$

"If a property holds about something, and an arbitrary instance of the property has some consequence, then the consequence holds in general. "

$\exists$ Elimination

Without the "c is not free in" side condition, $\exists$-E would be completely broken.

Q: Why is the following argument wrong?

| Someone is hungry.
| If Socrates is hungry, he eats a sandwich.
|———
| Therefore, Socrates eats a sandwich.

$\exists$ Elimination

Without the "c is not free in" side condition, $\exists$-E would be completely broken.

Q: Why is the following argument wrong?

Someone is hungry.
If Socrates is hungry, he eats a sandwich.
Therefore, Socrates eats a sandwich.

A: $\exists$-E does not apply, since Socrates is free in the conclusion of the argument on line 2.

About the online checker

The version of FOL supported by the online proof checker doesn't have function symbols: there's only constants and predicates.

The rules of natural deduction are the same whether function symbols are allowed or not.

Soundness

Theorem

Natural deduction is sound for Predicate Logic:

$$T \vdash \varphi \quad \textit{implies} \quad T \models \varphi$$

Soundness

Theorem

Natural deduction is sound for Predicate Logic:

$$T \vdash \varphi \quad \textit{implies} \quad T \models \varphi$$

Corollary

Natural deduction is consistent:

$$\emptyset \nvdash \bot$$

Completeness and Incompleteness

Confusingly, Kurt Gödel proved both of these:

Theorem (Gödel's completeness theorem)

Theorem (Gödel's first incompleteness theorem)

Completeness and Incompleteness

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Theorem (Gödel's completeness theorem)

Natural deduction is complete for Predicate Logic:

$$T \models \varphi \quad \textit{implies} \quad T \vdash \varphi$$

Theorem (Gödel's first incompleteness theorem)

Completeness and Incompleteness

Confusingly, Kurt Gödel proved both of these:

Theorem (Gödel's completeness theorem)

Natural deduction is complete for Predicate Logic:

$$T \models \varphi \quad \textit{implies} \quad T \vdash \varphi$$

Theorem (Gödel's first incompleteness theorem)

Roughly: if T is consistent, and expressive enough to do elementary arithmetic, then there are sentences G such that neither

$$T \vdash G \quad \textit{nor} \quad T \vdash \neg G$$

Summary of topics

- Well-formed formulas
- Boolean Algebras
- Valuations
- CNF/DNF
- Proof
- Natural deduction
- Bonus examples

Proof examples

What follows is a bunch of example derivations in natural deduction.

I do not plan to go over these in the lecture (favouring instead live demos).

Proof example (Fitch)

Prove: $\vdash \forall x \forall y\, (x = y) \to (y = x)$

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Prove: $\vdash \forall x \forall y\, (x = y) \rightarrow (y = x)$

1. a = b

Proof example (Fitch)

Prove: $\vdash \forall x \forall y\, (x = y) \rightarrow (y = x)$

1. a = b
2. a = a $\quad$ =-I

Proof example (Fitch)

Prove: $\vdash \forall x \forall y\, (x = y) \rightarrow (y = x)$

1. a = b	
2. a = a	=-I
3. b = a	=-E1: 1,2

Proof example (Fitch)

Prove: $\vdash \forall x \forall y\, (x = y) \to (y = x)$

1. $a = b$	
2. $a = a$	$=$-I
3. $b = a$	$=$-E1: 1,2
4. $(a = b) \to (b = a)$	$\to$-I: 1–3

Proof example (Fitch)

Prove: $\vdash \forall x \forall y\, (x = y) \to (y = x)$

1. a = b	
2. a = a	=-I
3. b = a	=-E1: 1,2
4. (a = b) → (b = a)	→-I: 1–3
5. ∀y (a = y) → (y = a)	∀-I: 4

Proof example (Fitch)

Prove: $\vdash \forall x \forall y\, (x = y) \rightarrow (y = x)$

1. a = b	
2. a = a	=-I
3. b = a	=-E1: 1,2
4. (a = b) → (b = a)	→-I: 1–3
5. ∀y (a = y) → (y = a)	∀-I: 4
6. ∀x∀y (x = y) → (y = x)	∀-I: 5

Proof example

Prove: $\forall x \forall y\, P(x, y) \vdash \forall y \forall x\, P(x, y)$

Line	Premises	Formula	Rule	References

Proof example

Prove: $\forall x \forall y\, P(x,y) \vdash \forall y \forall x\, P(x,y)$

Line	Premises	Formula	Rule	References
1		$\forall x \forall y\, P(x,y)$	Premise	

Proof example

Prove: $\forall x \forall y\, P(x, y) \vdash \forall y \forall x\, P(x, y)$

Line	Premises	Formula	Rule	References
1		$\forall x \forall y\, P(x, y)$	Premise	
2	1	$\forall y\, P(a, y)$	$\forall$-E	1

Proof example

Prove: $\forall x \forall y\, P(x,y) \vdash \forall y \forall x\, P(x,y)$

Line	Premises	Formula	Rule	References
1		$\forall x \forall y\, P(x,y)$	Premise	
2	1	$\forall y\, P(a,y)$	$\forall$-E	1
3	1	$P(a,b)$	$\forall$-E	2

Proof example

Prove: $\forall x \forall y\, P(x, y) \vdash \forall y \forall x\, P(x, y)$

Line	**Premises**	**Formula**	**Rule**	**References**
1		$\forall x \forall y\, P(x, y)$	Premise	
2	1	$\forall y\, P(a, y)$	$\forall$-E	1
3	1	$P(a, b)$	$\forall$-E	2
4	1	$\forall x\, P(x, b)$	$\forall$-I	3

Proof example

Prove: $\forall x \forall y\ P(x, y) \vdash \forall y \forall x\ P(x, y)$

Line	Premises	Formula	Rule	References
1		$\forall x \forall y\ P(x, y)$	Premise	
2	1	$\forall y\ P(a, y)$	$\forall$-E	1
3	1	$P(a, b)$	$\forall$-E	2
4	1	$\forall x\ P(x, b)$	$\forall$-I	3
5	1	$\forall y \forall x\ P(x, y)$	$\forall$-I	4

Proof example (Fitch)

Prove: $\exists x \exists y\, P(x, y) \vdash \exists y \exists x\, P(x, y)$

1. $\exists x \exists y \, \mathrm{P}(x, y)$

Proof example (Fitch)

Prove: $\exists x \exists y\, P(x, y) \vdash \exists y \exists x\, P(x, y)$

1. $\exists x \exists y\ \mathrm{P}(x, y)$
2. $\exists y\ \mathrm{P}(a, y)$

Proof example (Fitch)

Prove: $\exists x \exists y\, P(x,y) \vdash \exists y \exists x\, P(x,y)$

1. $\exists x \exists y\, \mathrm{P}(x,y)$

2. $\exists y\, \mathrm{P}(a,y)$

3. $\mathrm{P}(a,b)$

Proof example (Fitch)

Prove: $\exists x \exists y\, P(x, y) \vdash \exists y \exists x\, P(x, y)$

1. $\exists x \exists y\ \mathsf{P}(x, y)$
    2. $\exists y\ \mathsf{P}(a, y)$
        3. $\mathsf{P}(a, b)$
        4. $\exists x\ \mathsf{P}(x, b)$ $\exists$-I: 3

Proof example (Fitch)

Prove: $\exists x \exists y\, P(x,y) \vdash \exists y \exists x\, P(x,y)$

1. $\exists x \exists y\, \mathrm{P}(x,y)$
2. $\exists y\, \mathrm{P}(a,y)$
3. $\mathrm{P}(a,b)$
4. $\exists x\, \mathrm{P}(x,b)$ $\quad$ $\exists$-I: 3
5. $\exists y \exists x\, \mathrm{P}(x,y)$ $\quad$ $\exists$-I: 4

Proof example (Fitch)

Prove: $\exists x \exists y\, P(x, y) \vdash \exists y \exists x\, P(x, y)$

1. $\exists x \exists y\, P(x, y)$	
2. $\exists y\, P(a, y)$	
3. $P(a, b)$	
4. $\exists x\, P(x, b)$	$\exists$-I: 3
5. $\exists y \exists x\, P(x, y)$	$\exists$-I: 4
6. $\exists y \exists x\, P(x, y)$	$\exists$-E: 2, 3–5

Proof example (Fitch)

Prove: $\exists x \exists y\, P(x, y) \vdash \exists y \exists x\, P(x, y)$

1.	$\exists x \exists y\, P(x, y)$	
2.	$\exists y\, P(a, y)$	
3.	$P(a, b)$	
4.	$\exists x\, P(x, b)$	$\exists$-I: 3
5.	$\exists y \exists x\, P(x, y)$	$\exists$-I: 4
6.	$\exists y \exists x\, P(x, y)$	$\exists$-E: 2, 3–5
7.	$\exists y \exists x\, P(x, y)$	$\exists$-E: 1, 2–6